Lecture 24

Thm (Chevalley-Shephard-Todd) Let W be a finite subgroup of $GL(V)$ generated by reflections in codim-1 subspaces, pres no subsp.
Then 1) $S(V)^W$ is a poly ring w/ gens in different degrees $\underbrace{m_1 \dots m_r}_{\text{well-def}}$. (gens are not!)
2) $S(V)$ is a free module over $S(V)^W$

Nontrivial! e.g. $\mathbb{R}^2 = V$, $W = \langle -I \rangle$ then $S(V) \cong \mathbb{R}[x,y]$
$S(V)^W = \mathbb{R}[x^2, y^2]$ $\quad S(V) \cong S(V)^W[\varepsilon, \delta] / (\varepsilon^2, \delta^2)$

Proof also shows $\prod m_i = |W| = \text{rank}(S(V)$ as module over $S(V)^W)$

Ex. Sym_n, $\varepsilon_1 \cdots \varepsilon_n$, $m_i = i$, $i = 1..n$. $\prod m_i = n!$

$\mathcal{R} = S(V)/(S(V)^W_+)$ For $V = \mathfrak{t}^*$, $\mathcal{R}_1 = \mathfrak{t}^*$ because there are no inv't subsp

Thm (Borel): $\exists$ natural degree-doubling ring iso

$$cw : \mathcal{R} \longrightarrow H^*(G/B, \mathbb{R})$$

Cor H^* gen as an algebra by H^2, and if the thm is true, cw uniquely det by its restr to $\mathcal{R}_1 \cong \mathfrak{t}^*$.

Let $\lambda \in \mathfrak{t}^*$ Get $\lambda : \mathfrak{h} \to \mathbb{C}$ linear $\Rightarrow$ $\tilde{\lambda} : H \to \mathbb{C}^*$ grp hom — assume $\lambda(\ker \exp_{\mathfrak{t}}) \subset 2\pi\mathbb{Z}$
$\tilde{\lambda}(\exp x) = e^{\lambda(x)}$
$B = HU$ retracts onto H: $hu \mapsto h$ is well-defined
Compose this retr w/ $\tilde{\lambda}$, get $\tilde{\lambda} : B \to \mathbb{C}^*$ grp hom.
Now take the product $G \times \mathbb{C}$. B acts on this on right by

$$(g, z) \cdot b = (gb, \tilde{\lambda}(b) z)$$

Let $\mathcal{L}_\lambda$ denote the quotient. Also called $G \times_\lambda \mathbb{C}$

The map $G \times \mathbb{C} \to G$ descends to a map $\tilde{\mathcal{L}} \to G/B$.
This is a locally trivial fibration w/ fibers $\mathbb{C}$.
$\mathcal{L}_\lambda$ is a holo line bdl / invertible sheaf.

$\mathcal{U} \subset G/B$ $\quad \mathcal{L}_\lambda(\mathcal{U}) =$ maps $\pi^{-1}(\mathcal{U}) \xrightarrow{f} \mathbb{C}$ that are holo and B equiv't
$\pi : G \to G/B$ $\quad$ i.e. $\quad f(gb) = \tilde{\lambda}(b) f(g)$.

Def. CW is the ring hom def by, for $\lambda \in \mathcal{R}_1 \cong \mathfrak{t}^*$
$cw(\lambda) = c_1(\mathcal{L}_\lambda)$ where c_1 is the first Chern class.

HELP, I DON'T KNOW ABOUT CHERN CLASSES!

No problem. Here are two ways to look at them.

① Spse you want to describe a hypersurface in G/B by a global eqn like $f = 0$. You can't expect f to be holo.
G/B is compact so f is constant.

Instead, we look at G and try to find holo fns $G \to \mathbb{C}$ where $f^{-1}(0)$ is B-inv't, so descends to subset G/B.

Oops, we're back to the same problem! Holo on G/B.

Fix: If $f : G \to \mathbb{C}$ just has $f(gb) =$ some nonzero multiple of $f(g)$ then it's still true that $f^{-1}(0)/B$ is well-def.
The multiplier gives a hom $\tilde{\lambda} : B \to \mathbb{C}^*$.
Then f actually gives a section $G/B \to \mathcal{L}_\lambda$
$gB \mapsto [(g, f(g))]$

Almost. $c_1(\mathcal{L}_\lambda) = [f^{-1}(0)/B]$ if f is such a fn.
fund class

Actual. f may not exist. Must allow mero.
Then $f^{-1}(0)/B$ can be written as $\bigcup_i X_i$ $\qquad X_i \subset G/B$ irred subvar. codim 1
Let $n_i = \mathrm{ord}_f X_i$

$c_1(\mathcal{L}_\lambda) = \sum_i n_i [X_i]$.

<u>Ex.</u> $SL_2\mathbb{C}$. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto a$.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & t \\ 0 & \frac{1}{x} \end{pmatrix} = \begin{pmatrix} ax & at + \frac{b}{x} \\ cx & ct + d/x \end{pmatrix} \longmapsto ax = \tilde{\lambda}\begin{pmatrix} x & t \\ 0 & \frac{1}{x} \end{pmatrix}$$

where $\lambda = \frac{1}{2}\alpha$

$\alpha\begin{pmatrix} t & \\ & -t \end{pmatrix} = 2t$. $\lambda\begin{pmatrix} t & \\ & -t \end{pmatrix} = t$.

$SL_2\mathbb{C}/B \cong \mathbb{CP}^1$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} B \longmapsto [a:c]$.

$c_1(\mathcal{L}_\lambda) = [pt] = $ gen of $H^2(\mathbb{CP}^1, \mathbb{Z})$

$c_1(\mathcal{L}_\alpha) = 2 \cdot [pt]$.

<u>Ex.</u> $SL_3\mathbb{C}$ case!

② Universality.

Topo line bundle: Bundle w/ fiber $\mathbb{C}$ (continuous local triv) and where gluing $U \times \mathbb{C}$ to $V \times \mathbb{C}$ happens using fiberwise linear map $U \cap V \to \mathbb{C}^* = GL_1(\mathbb{C})$.

There is a space and a line bundle over that space

$$\begin{array}{c} \mathbb{L} \\ \downarrow \\ \mathbb{B} \end{array}$$

s.t $\forall$ $\mathcal{L} \to X$ line bdl, $\exists!$ up to htpy map $f: X \to \mathbb{B}$

s.t. $\mathcal{L} \cong f^*(\mathbb{L})$

So then any class in $H^*(\mathbb{B})$ gives an invt of line bdls.

$\mathcal{L} \longmapsto f^*(\alpha)$ $\alpha \in H^*(\mathbb{B})$.

Here's the best part: $\mathbb{B}$ is htpy equiv to $\mathbb{CP}^\infty$

so $H^*(\mathbb{CP}^\infty) \cong \mathbb{R}[x]$ w/ x in degree 2.

Then $c_1(\mathcal{L}) = f_\chi^*(x)$!